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Transformation of Boundary Value Problems into Initial Value Problems

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In a previous paper of the author (SIAM Review, 1962, pp. 43–47), the idea of Blasius for the transformation of the B.V.P., $y''' + yy'' = 0, y(0) = y'(0) = 0$, $y'(\infty) = 2$, into a pair of similar I.V.P.'s was extended to D.E.'s or systems of D.E.'s of any order which were invariant under certain groups of homogeneous linear transformations. The boundary conditions were specified at the origin and at infinity and were homogeneous at the initial point. Subsequently, T. Y. Na (SIAM Review, 1967, pp. 204–210; 1968, pp. 85–87) showed that the method was also applicable to finite intervals and also considered other groups of transformations. He noted that for the method to apply, the boundary conditions have to be homogeneous at the initial point. Here, however, we show that homogeneity is not necessary by treating equations subject to a variety of nonhomogeneous boundary conditions.

1. INTRODUCTION

In a previous paper of the author [1], the idea of Blasius [2] for the transformation of the boundary value problem[1] (B.V.P.)

$$y''' + yy'' = 0,$$
$$y(0) = y'(0) = 0, \qquad y'(\infty) = 2,$$

into the pair of initial value problems (I.V.P.'s)

$$\begin{gathered} F''' + FF'' = 0, \\ F(0) = F'(0) = 0, \qquad F''(0) = 1, \end{gathered} \tag{A}$$

$$\begin{gathered} y''' + yy'' = 0, \\ y(0) = y'(0) = 0, \qquad y''(0) = \left\{\frac{2}{F'(\infty)}\right\}^{3/2}, \end{gathered} \tag{B}$$

[1] This problem results from making a similarity transformation in the relevant partial differential equations for the steady two-dimensional flow along a flat plate placed edgewise to a stream.

was extended in several ways.[2] The methods were applicable to ordinary differential equations or systems of ordinary differential equations which were invariant under certain groups of homogeneous linear transformations. Additionally, the boundary conditions were specified at the origin and at infinity and were homogeneous at the initial point. Subsequently, in two papers by Na [4, 5], it was shown that method was applicable to finite intervals and also to equations which were invariant under other groups of transformations. It was also stated [4, p. 204] that the boundary conditions (B.C.'s) at the initial point must be homogeneous at the initial point for the method to be applicable. Although all the B.C.'s considered in [1], [4], and [5] were homogeneous at the initial point, it will be shown subsequently that this condition is not always necessary. Additionally, an error in [4, p. 207] will be pointed out and corrected.

In all the B.V.P.'s and I.V.P.'s to be considered, we are tacitly assuming the existence and uniqueness[3] of the latter which in turn will have implications for the former. We will return to this point subsequently.

2. Second-Order B.V.P.'s

First, we consider the rather general second-order differential equation

$$\sum_{m,n,r,s} A_{mnrs} y''^m y'^n y^r x^s = 0 \tag{1}$$

subject to the boundary conditions (which are more general than those treated in [1], [4], [5])

$$y'(0) = ay(0) + b, \qquad y^{(e)}(\infty) = k. \tag{2}$$

Here m, n, r, s are arbitrary indices, A_{mnrs} are arbitrary constants and (e) is an arbitrary integer. If (1) is multivalued for y'', we assume a particular branch is specified.

We now assume that y can be expressed in the form

$$y = \lambda F(\mu x) \tag{3}$$

[2] In principle, there is no need to solve the second I.V.P. (B), since $F(x)$ has been determined from (A) and $y = \lambda^{1/3} F(\lambda^{1/3} x)$ where $2 = \lambda^{2/3} F'(\infty)$. However, if y is to be given at the same uniformly spaced values of x as $F(x)$, then it is probably easier and more accurate to solve (B) than to interpolate the values of $\lambda^{1/3} F(\lambda^{1/3} x)$ from $F(x)$. The problem of how large x has to be to approximate to $x = \infty$ is treated by Rubel [3].

[3] A simple example of the nonuniqueness of an I.V.P. is given by $y'' = 20y^{3/5}$, $y(0) = y'(0) = 0$. An infinite class of solutions is given by $y = 0$, $0 \leqslant t \leqslant t_0$, $y = (t - t_0)^5$, $t \geqslant t_0$. This is an example of unstable motion and is related to incorrect "physical proofs" (still occurring in present day texts) that the motion of a particle under a central force is coplanar [6].

where $F(x)$ also satisfies (1) for arbitrary constants λ, μ but subject to the initial conditions

$$F(0) = F'(0) = 1.$$

In order that both y and $F(x)$ satisfy (1), the D.E. must be invariant under the two-parameter group of homogeneous linear transformations

$$x_1 = \mu x, \qquad y_1 = \frac{y}{\lambda}.$$

The condition that this implies on the indices m, n, r, s is gotten by substituting (3) into (1), i.e.,

$$\sum_{m,n,r,s} A_{mnrs} F''(\mu x)^m F'(\mu x)^n F(\mu x)^r (\mu x)^s \lambda^c \mu^d = 0,$$

where

$$c = m + n + r, \qquad d = 2m + n - s. \tag{4}$$

Consequently, c and d must be constant for all sets of indices m, n, r,s and then (1) reduces to

$$\sum_{m,n} A_{mn} y''^m y'^n y^r x^s = 0, \tag{5}$$

where r and s are given by (4). It now follows that

$$\begin{gathered} y(0) = \lambda, \qquad y'(0) = a\lambda + b = \lambda\mu, \\ k = \lambda\mu^e F^{(e)}(\infty). \end{gathered} \tag{6}$$

We now solve the initial value problem for F and determine $F^{(e)}(\infty)$. Then λ and μ are determined from the simultaneous equations in (6) which then give $y(0)$ and $y'(0)$. Thus we have converted the original B.V.P. into two similar I.V.P.'s. Or else as noted in footnote 2, y can be determined from $y = \lambda F(\mu x)$.

Having tacitly assumed the existence and uniqueness of $F(x)$ in $(0, \infty)$, the existence and uniqueness of y depends on the existence and uniqueness of μ and λ. Eliminating λ in (6) gives

$$\mu^e = k'(\mu - a).$$

Thus, depending on the relative values of e, k' and a, there can be zero, one, two or three solutions for μ and the same correspondingly for y.

The case for a finite interval instead of an infinite interval for the preceding problem can be treated in a similar fashion. If the second B.C. was given by

$$y^{(e)}(L) = k, \tag{2'}$$

we would then have

$$y'(0) = a\lambda + b = \lambda\mu, \qquad k = \lambda\mu^e F^{(e)}(\mu L).$$

Again there are two equations for λ and μ but not quite so simple as before. Although these equations can be easily solved numerically, it may be easier to start out with the original system, pick a trial value for λ and then refine it by interpolation techniques. For higher order equations, the method will be more useful.

If instead of B.C.'s (2), we had $y(0) = a$, $y^{(e)}(\infty) = k$, we would assume that y could be expressed in the form $y = F(\mu x)$ where $F(x)$ also satisfies (1) for arbitrary μ. This entails that in (1)

$$2m + n - s = \text{const.}$$

Since $y(0) = a$, $F(0) = a$. If we now let $F'(0) = 1$, then $y'(0) = \mu$, which is determined from

$$k = \mu^e F^{(e)}(\infty)$$

provided $e \neq 0$. For the case $e = 0$, we have an anomaly that could be due to impossible B.C.'s. As an example, consider the D.E.

$$[xD - 1]\,[xD + 1]\,y = 0$$

whose solution is $y = Ax + B/x$.

For a finite interval, with B.C. $y^{(e)}(L) = k$ instead of $y^{(e)}(\infty) = k$, we can proceed as before. We could also replace the B.C. at the terminal point by either of the more general ones

$$\sum_i A_i y^{(i)}(L) = k \qquad \text{or} \qquad \lim_{x\to\infty} \sum_i A_i x^{e_i} y^{(i)}(x) = k.$$

3. Third-Order B.V.P.'s

We now consider third-order equations of the form,

$$\sum_{m,n,r,s,t} A_{mnrst}\, y'''^m y''^n y'^r y^s x^t = 0. \tag{7}$$

In [5, p. 207], Na considers B.C.'s of the forms

Case 1.

$$y(0) = 0, \qquad y^{(d_i)}(\infty) = k_i \qquad (i = 1, 2),$$

Case 2.

$$y(o) = 0, \qquad y^{(d_1)}(L) = 0, \qquad y^{(d_2)}(L) = k_2,$$

and then defines the two-parameter group of transformations,

$$x = B^{\beta_1}C^{\gamma_1}\bar{x}, \qquad y = B^{\beta_2}C^{\gamma_2}\bar{y}.$$

However, the method as used does not work. Although a two parameter group of transformations is set up, the way that it is subsequently used is equivalent to starting with a one parameter group. Since his Eqs. (22) and (23) are equivalent, they will lead to

$$\frac{\beta_2}{\beta_1} = \frac{\gamma_2}{\gamma_1} = a.$$

Then his two Eqs. (27) and (28) for Case 1 which are to be used for determining B and C reduce to

$$\{B^{\beta_1}C^{\gamma_1}\}^{a-d_1} = \frac{k_1}{\bar{y}^{(d_1)}(\infty)},$$

$$\{B^{\beta_1}C^{\gamma_1}\}^{a-d_2} = \frac{k_2}{\bar{y}^{(d_2)}(\infty)}.$$

In general, the latter two equations will be inconsistent and similarly for Case 2.

A proper and simpler way to proceed is to assume that y can be expressed in the form given by (3). This entails that $(m + n + r + s)$ and $(3m + 2n + r - t)$ must both be constant for all sets of indices m, n, r, s. Then $F(o) = 0$, and letting

$$F'(o) = F''(o) = 1,$$

we get that

$$y'(o) = \lambda\mu, \qquad y''(o) = \lambda\mu^2$$

where λ and μ are determined from

$$k_1 = \lambda\mu^{d_1}F^{(d_1)}(\infty),$$

$$k_2 = \lambda\mu^{d_2}F^{(d_2)}(\infty),$$

which could be solved numerically for μ after eliminating λ.[4] Again as noted

[4] And similarly for a finite interval.

in the previous example, the existence and uniqueness of y will depend on the corresponding existence and uniqueness for μ.

Incidentally, the B.C.'s specified by Na in case (1) for an infinite interval are unrealistic. For if $y^{(d_1)}(\infty) = k_1$, then usually $y^{(d_2)}(\infty) = 0$ for $d_2 > d_1$. Consequently, the given B.C.'s should be replaced by something of the type

$$\lim_{x\to\infty} x^{e_i} y^{(d_i)}(x) = k_i, \qquad i = 1, 2.$$

Then the determining equations for λ and μ will be

$$\lambda\mu^{d_i - e_i} = \lim_{t\to\infty} \frac{k_i}{t^{e_i} F^{(d_i)}(t)}, \qquad i = 1, 2,$$

provided $\mu > 0$. If $\mu < 0$, $t \to \infty$ is replaced by $t \to -\infty$.

We now list some of the other sets of B.C.'s for (7) which can be treated in a similar manner. The indices m, n, r, s of (7) will have to satisfy conditions such that the indicated $F(x)$ is also a solution. In each case the condition at $x = L$ can be replaced by the same condition at $x = \infty$ provided they are meaningful. If not, they can be replaced with the type of conditions just discussed previously.

$$\begin{gathered} y(0) = 0, \qquad y'(0) = a, \qquad \sum_i a_i y^{(i)}(L) = b; \\ y(x) = \lambda^{-1} F(\lambda x), \\ F(0) = 0, \qquad F'(0) = a, \qquad F''(0) = 1. \end{gathered} \tag{8}$$

$$\begin{gathered} y(0) = a, \qquad y'(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = b; \\ y(x) = F(\lambda x), \\ F(0) = a, \qquad F'(0) = 0, \qquad F''(0) = 1. \end{gathered} \tag{9}$$

$$\begin{gathered} y'(0) = hy(0) + k, \qquad y''(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = b; \\ y(x) = \lambda F(\mu x), \\ F(0) = 1, \qquad F'(0) = 1, \qquad F''(0) = 0. \end{gathered} \tag{10}$$

$$\begin{gathered} y(0) = 0, \qquad \sum_i a_{ij} y^{(e_{ij})}(L) = b_j \qquad (j = 1, 2); \\ y(x) = \lambda F(\mu x), \\ F(0) = 0, \qquad F'(0) = 1, \qquad F''(0) = 1. \end{gathered} \tag{11}$$

$$y'(0) = 0, \qquad \sum_i a_{ij} y^{(i)}(L) = b_j \qquad (j = 1, 2);$$
$$y(x) = \lambda F(\mu x), \tag{12}$$
$$F(0) = 1, \qquad F'(0) = 0, \qquad F''(0) = 1.$$

$$y''(0) = 0, \qquad \sum_i a_{ij} y^{(i)}(L) = b_j \qquad (j = 1, 2);$$
$$y(x) = \lambda F(\mu x), \tag{13}$$
$$F(0) = 1, \qquad F'(0) = 1, \qquad F''(0) = 0.$$

For fourth-order equations of the form,

$$\sum A_{mnrstu} y''''^m y'''^n y''^r y'^s y^t x^u = 0, \tag{14}$$

some of the B.C.'s which can be treated are listed as follows:

$$y(0) = y'(0) = y''(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = b;$$
$$y = \mu^\alpha F(\mu x); \tag{15}$$
$$F(0) = F'(0) = F''(0) = 0, \qquad F'''(0) = 1.$$

$$y(0) = a, \qquad y'(0) = y''(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = b;$$
$$y = F(\mu x) \tag{16}$$
$$F(0) = a, \qquad F'(0) = F''(0) = 0, \qquad F'''(0) = 1.$$

$$y(0) = y''(0) = 0, \qquad y'(0) = a, \qquad \sum_i a_i y^{(i)}(L) = b;$$
$$y = \mu^{-1} F(\mu x), \tag{17}$$
$$F(0) = F''(0) = 0, \qquad F'(0) = F'''(0) = 1$$

$$y(0) = y'(0) = 0, \qquad \sum_i a_{ij} y^{(i)}(L) = b_j \qquad (j = 1, 2);$$
$$y = \lambda F(\mu x), \tag{18}$$
$$F(0) = F'(0) = 0, \qquad F''(0) = F'''(0) = 1.$$

$$y'(0) = hy(0) + k, \qquad y''(0) = y'''(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = b;$$
$$y = \lambda F(\mu x), \tag{19}$$
$$F(0) = F'(0) = 1, \qquad F''(0) = F'''(0) = 0.$$

We can also treat similar type equations of any order subject to similar type B.C.'s.

4. Simultaneous B.V.P.'s

In [1], it was also shown how to treat systems of simultaneous equations by means of a typical example of a system of two second-order equations in y and z. Here we will show how to treat such equations subject to a broader class of B.C.'s by means of one illustrative example of two simultaneous equations, one of third order in y and second-order in z. This particular class of equations arises in a number of applications. For example, in their study of the flow of a viscous, electrically conducting incompressible fluid past a semiinfinite plate in the presence of a magnetic field, Greenspan and Carrier [7] have shown that the boundary layer equations may be reduced to the form,

$$f''' + ff'' - \beta gg'' = 0, \qquad g'' + \epsilon(fg' - f'g) = 0,$$

subject to the boundary conditions,

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(\infty) = 2,$$
$$g(0) = 0, \qquad g'(\infty) = 2.$$ [5]

Although the above equations are invariant under the one parameter group of transformations

$$x_1 = \lambda x, \qquad f_1 = \frac{f}{\lambda}, \qquad g_1 = \frac{g}{\lambda},$$

this is not sufficient to effect a transformation of the B.V.P. into an I.V.P. (since there are two conditions specified at infinity). A twoparameter group of transformations is necessary but unfortunately no two parameter groups of homogeneous linear transformations exist. This does not rule out other possible types of transformations.

Let us now consider the system

$$\begin{aligned} &\sum_i A_i y_3^{m_i} y_2^{n_i} y_1^{p_i} y^{q_i} z_2^{r_i} z_1^{s_i} z^{t_i} x^{u_i} = 0, \\ &\sum_i \bar{A}_i y_3^{\bar{m}_i} y_2^{\bar{n}_i} y_1^{\bar{p}_i} y^{\bar{q}_i} z_2^{\bar{r}_i} z_1^{\bar{s}_i} z^{\bar{t}_i} x^{\bar{u}_i} = 0, \end{aligned} \tag{20}$$

(here $y_3 = d^3y/dx^3$, etc.), subject to the B.C.'s

$$y(0) = a, \qquad y'(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = k$$

$$z(0) = 0, \qquad \sum_i b_i z^{(i)}(L) = l.$$

[5] In an interesting note, Reuter and Stewartson [8] show that there can be no solutions if $\beta > 1$.

Since there is one nonhomogeneous B.C. at the initial point and two B.C.'s at the terminal point, we need a three-parameter group of transformations. Consequently, we assume that

$$y = \lambda F(\mu x), \qquad z = \eta G(\mu x) \tag{21}$$

where λ, η, and μ are arbitrary parameters such that F and G also satisfy (20). On substituting back, this imposes the following six sets of conditions for each value of the index i:

$$\begin{aligned} m_i + n_i + p_i + q_i &= \text{const}, \\ \bar{m}_i + \bar{n}_i + \bar{p}_i + \bar{q}_i &= \text{const}, \\ r_i + s_i + t_i &= \text{const}, \\ \bar{r}_i + \bar{s}_i + \bar{t}_i &= \text{const}, \\ 3m_i + 2n_i + p_i + 2r_i + s_i - u_i &= \text{const}, \\ 3\bar{m}_i + 2\bar{n}_i + \bar{p}_i + 2\bar{r}_i + \bar{s}_i - \bar{u}_i &= \text{const}. \end{aligned} \tag{22}$$

If we now let

$$F(0) = 1, \qquad F'(0) = 0, \qquad F''(0) = 1,$$
$$G(0) = 0, \qquad G'(0) = 1,$$

then

$$a = \lambda, \qquad y''(0) = \lambda\mu^2, \qquad z'(0) = \eta\mu,$$
$$\lambda \sum_i a_i \mu^i F^{(i)}(\mu L) = k, \qquad \eta \sum_i b_i \mu^i G^{(i)}(\mu L) = l.$$

In general, we can determine μ and then η from the latter equations. Then y and z are given by (21). Again, we can replace the finite interval by an infinite interval.

If the B.C. $y(0) = a$ was replaced by $y(0) = 0$, then in the above we would let $\lambda = 1$ and change $F(0) = 1$ to $F(0) = 0$ and keep everything else the same. For this case, however, we can get by with less restrictions on the indices as given by (22). All we need now is a two-parameter group of transformations. Consequently, we assume that

$$y = \lambda F(\mu x), \qquad z = \lambda^\alpha \mu^\beta G(\mu x),$$

where λ, μ are arbitrary parameters, α and β are constants to be determined, such that F and G satisfy (20). This only imposes the following four sets of conditions for each i plus the additional advantage of having two extra constants (α and β) at our disposal:

$$\begin{aligned}
m_i + n_i + p_i + q_i + \alpha(r_i + s_i + t_i) &= \text{const},\\
\bar{m}_i + \bar{n}_i + \bar{p}_i + \bar{q}_i + \alpha(\bar{r}_i + \bar{s}_i + \bar{t}_i) &= \text{const},\\
3m_i + 2n_i + p_i + (\beta + 2)\, r_i + (\beta + 1)\, s_i + \beta t_i - u_i &= \text{const},\\
3\bar{m}_i + 2\bar{n}_i + \bar{p}_i + (\beta + 2)\, \bar{r}_i + (\beta + 1)\, \bar{s}_i + \beta \bar{t}_i - \bar{u}_i &= \text{const}.
\end{aligned}$$

As before, we only list some of the other sets of B.C.'s for (20) which can be treated in a similar manner. The indices of (20) will have to satisfy the appropriate conditions such that the indicated $F(x)$ and $G(x)$ are also solutions. In each case the condition at $x = L$ can be replaced by the same condition at $x = \infty$

$$\begin{gathered}
y'(0) = ay(0) + b, \qquad y''(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = k,\\
z(0) = 0, \qquad \sum_i b_i z^{(i)}(L) = l;\\
y = \lambda F(\mu x), \qquad z = \eta G(\mu x),\\
F(0) = 1, \qquad F'(0) = 1, \qquad F''(0) = 0,\\
G(0) = 0, \qquad G'(0) = 1.
\end{gathered} \tag{23}$$

$$\begin{gathered}
y(0) = 0, \qquad y'(0) = 0, \qquad \sum_i a_1 y^{(i)}(L) = k,\\
z(0) = a, \qquad \sum_i b_i z^{(i)}(L) = l;\\
y = \lambda F(\mu x), \qquad z = \eta G(\mu x)\\
F(0) = 0, \qquad F'(0) = 0, \qquad F''(0) = 1,\\
G(0) = 1, \qquad G'(0) = 1.
\end{gathered} \tag{24}$$

$$\begin{gathered}
y(0) = 0, \qquad y'(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = k,\\
z(0) = a, \qquad z'(0) = b;\\
y = \lambda F(\mu x), \qquad z = \eta G(\mu x),\\
F(0) = 0, \qquad F'(0) = 0, \qquad F''(0) = 1,\\
G(0) = 1, \qquad G'(0) = 1.
\end{gathered} \tag{25}$$

$$\begin{gathered}
y(0) = a, \qquad y'(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = k,\\
z(0) = 0, \qquad z'(0) = b;\\
y = \lambda F(\mu x), \qquad z = \eta G(\mu x),\\
F(0) = 1, \qquad F'(0) = 0, \qquad F''(0) = 1,\\
G(0) = 0, \qquad G'(0) = 1.
\end{gathered} \tag{26}$$

$$
\begin{gathered}
y(0) = 0, \qquad y'(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = k, \\
z(0) = 0, \qquad z'(0) = b, \\
y = \eta^\alpha \mu^\beta F(\mu x), \qquad z = \eta G(\mu x), \\
F(0) = 0, \qquad F'(0) = 0, \qquad F''(0) = 1, \\
G(0) = 0, \qquad G'(0) = 1.
\end{gathered} \tag{27}
$$

$$
\begin{gathered}
y(0) = az'(0) + b, \qquad y'(0) = 0, \qquad \sum_i a_i y^{(i)}(L) = k, \\
z(0) = 0, \qquad \sum_i b_i z^{(i)}(L) = l; \\
y = \lambda F(\mu x), \qquad z = \eta G(\mu x), \\
F(0) = 1, \qquad F'(0) = 0, \qquad F''(0) = 1, \\
G(0) = 0, \qquad G'(0) = 1.
\end{gathered} \tag{28}
$$

$$
\begin{gathered}
y(0) = az(0) + b, \qquad y'(0) = 0, \qquad y''(0) = 0, \\
z'(0) = 0, \qquad H(y^{(e)}(L), z^{(f)}(L)) = 0; \\
y = \lambda F(\mu x), \qquad z = \lambda^\alpha \mu^\beta G(\mu x), \\
F(0) = 1, \qquad F'(0) = 0, \qquad F''(0) = 0, \\
G(0) = 1, \qquad G'(0) = 0.
\end{gathered} \tag{29}
$$

The method can also be used on similar systems of equations of any order and any number of dependent variables subject to a variety of similar type B.C.'s. For the method to apply, the system of equations will have to be invariant under a group of transformations with an appropriate number of parameters. For homogeneous linear transformations, this number will correspond to the sum of the number of conditions specified at the terminal point plus the number of nonhomogeneous or mixed conditions at the initial point. The condition $y'(0) = ay(0)$ is considered a mixed one since it contains more than one derivative (the zeroth and the first). Although this condition is homogeneous, it still requires an extra parameter.

5. Transformation of B.C.'s

There are B.V.P.'s where the previous method will not apply directly unless the B.C.'s are first transformed into a suitable form. As an example, consider the B.V.P. [9]

$$
x^{n/(n+1)} y'' = y^{(2n+1)/(n+1)}, \qquad y(0) = 1, \qquad y(\infty) = 0. \tag{30}
$$

For $n = 1$, we get the Thomas-Fermi equation [10] which had arisen in the determination of the effective nuclear charge in heavy atoms.

It follows that if $F(x)$ is a solution of the above D.E., so also is

$$y = \lambda^{(n+2)/n}F(\lambda x).$$

Even though the equation has a one parameter group of transformations, the method does not work. It would work, if the B.C.'s were interchanged to

$$y(0) = 0, \qquad y(\infty) = 1.$$

This can be done by letting $x \to 1/x$. Since

$$y'' \to x^4 y'' + 2xy'$$

it might appear as if we will lose the one parameter group of transformations. Fortunately, if for any D.E. where $F(x)$ and $\lambda^\alpha F(\lambda x)$ are both solutions, one makes the transformation $x \to x^{-r}$, then if $G(x)$ is a solution so also is $\lambda^\alpha G(x/\lambda^{1/r})$. Similar results apply if we transform $y \to y^s$.

Carrying out the transformation $x \to 1/x$ on (30), we get

$$\begin{aligned} x^{-n/(n+1)}\{x^4 y'' + 2x^3 y'\} &= y^{(2n+1)/(n+1)}, \\ y(0) = 0, \qquad y(\infty) &= 1. \end{aligned} \tag{31}$$

Now letting $y = \lambda^{(n+2)/n}F(x/\lambda)$ with $F(0) = 0$, $F'(0) = 1$, we get that

$$\lambda = \{F(\infty)\}^{-n/(n+2)}.$$

It should be noted that the D.E. is singular at the origin. Consequently, in order to start the numerical solution for the I.V.P., one will first have to find the asymptotic solution in the neighborhood of the origin.

6. Non-Linear Groups of Transformations

As an example which can be treated by groups of transformation other than linear, Na [4, p. 209] considers the following B.V.P. for steady heat conduction with heat generation which is given in Carslaw [11]:

$$\begin{aligned} \frac{d^2T}{dx^2} + \beta e^T &= 0, \\ T'(0) = 0, \qquad T(1) &= 0. \end{aligned} \tag{32}$$

In slightly different form, Na shows that if $T = F(x)$ is a solution of (32), so also is

$$T = F(e^\alpha x) + 2\alpha.$$

Then if $T(o) = 2\alpha$, $F(o) = 0$. Also, $F'(o) = 0$. After solving F as an I.V.P., α is gotten from $T(1) = 0$, or equivalently,

$$F(r) = -2 \log r \tag{33}$$

where

$$\alpha = \log r.$$

Whether or not a solution exists for (32) depends on the existence of a solution for (33). Since (32) is actually integrable, it has been shown [11] that there are two solutions if $0 < \beta < 0.88$ and none if $\beta > 0.88$. This was verified for various values of β very quickly by solving the I.V.P. for F, using the simple Euler method, on a time sharing console just using "basic" programming.[6] Initially, for each value of β, we had to carry out its corresponding integration. Subsequently, it was realized that only one integration was necessary for a given set of values of β. This follows since if $F(x)$ is a solution of

$$\frac{d^2T}{dx^2} + e^T = 0,$$

then

$$T = F(xe^{\alpha}\beta^{1/2}) + 2\alpha$$

is a solution of (32). Then choosing,

$$F(o) = F'(o) = 0,$$

we have

$$F(e^{\alpha}\beta^{1/2}) = -2\alpha$$

or

$$F(s) = -2 \log s\beta^{-1/2},$$

where $s = e^{\alpha}\beta^{1/2}$. Consequently, we only have to numerically integrate the I.V.P. for $F(x)$ and have the computer print out $F(x) + 2 \log x\beta^{-1/2}$ whenever it changes sign for a given set of values of β [which will then give the corresponding value of $T(o)$].

If in (32), β is replaced by βx^n, we still could determine the range of β necessary for existence numerically as before. Here, T and F would now be related by

$$T = F(xe^{\alpha}\beta^{1/(n+2)}) + 2\alpha.$$

[6] An example with more numerical details is given in the Appendix.

In [5], Na extends the applications of spiral groups of transformations to the more general class of B.V.P.'s,

$$\sum_i C_i y_2^{m_i} y_1^{n_i} e^{p_i y} x^{q_i} = 0, \tag{34}$$

subject to the boundary conditions

$$y'(\mathrm{o}) = y(1) = 0, \qquad \text{or} \qquad y'(\mathrm{o}) = 0, \qquad y(\infty) = k; \tag{35}$$

$$\sum_i A_i y_3^{m_i} y_2^{n_i} y_1^{r_i} e^{s_i y} x^{t_i} = 0$$

subject to (presumably) the same boundary conditions as in Cases 1 and 2.

If (34) is to be invariant under the group of spiral transformations

$$\bar{y} = y - \beta, \qquad \bar{x} = e^{\alpha} x,$$

then the indices of (34) must satisfy

$$\alpha(2m_i + n_i - q_i) + \beta p_i = \text{const.}$$

Since β is a function of α, the transformation is a one-parameter group and if $F(x)$ satisfies (34), so will

$$y(x) = F(e^{\alpha} x) + \beta(\alpha).$$

It now follows that the B.C.'s $y'(\mathrm{o}) = y(1) = 0$ can be extended to the somewhat more general set

$$y'(\mathrm{o}) = 0, \qquad \sum_i a_i y^{(i)}(L) = k.$$

For letting $y(0) = \beta(a)$, we get that $F(\mathrm{o}) = 0, F'(\mathrm{o}) = 0$. Then α is determined from

$$k = \sum_i a_i e^{\alpha i} F^{(i)}(Le^{\alpha}) + a_0 \beta(\alpha).$$

Additionally, we can treat B.C.'s of the type

$$y(\mathrm{o}) = k, \qquad \lim_{x \to \infty} x^2 y'(x) = 0.$$

Since (34) is invariant under a spiral group of transformations, it will continue to be after the transformation $x \to x^{-r}$. For the case $r = 1$, $\beta \to -\beta$ and the new B.C.'s are

$$y'(\mathrm{o}) = 0, \qquad y(\infty) = k$$

which has already been treated.

If the B.C.'s for (35) are as indicated, they are unrealistic as was noted previously. However, even with these B.C.'s, the method (as is briefly indicated) will not work since Na's ostensible two-parameter spiral group of transformations

$$x = e^{\beta_1 B + \gamma_1 C}\bar{x}, \qquad y = \bar{y} + \beta_2 B + \gamma_2 C$$

really ends up as a one-parameter group as for (34) (we are assuming that at least one $s_i \neq 0$).

We now give a set of B.C.'s which will be resolved by means of the *one*-parameter group. Assume that both $y(x)$ and $F(x)$ are solutions of (35), where

$$y(x) = F(e^{\alpha}x) + \beta.$$

This requires that

$$\alpha(3m_i + 2n_i + r_i + t_i) + \beta s_i = \text{const}$$

and gives β as a function of α. If

$$y'(0) = 0 = y''(0), \qquad \sum_i a_i y^{(i)}(1) = k,$$

we let $y(0) = \beta(\alpha)$. Then,

$$F(0) = F'(0) = F''(0) = 0,$$

and α is determined from

$$\sum_i a_i e^{\alpha i} F^{(i)}(e^{\alpha}) + a_0 \beta(\alpha) = k.$$

Some other sets of B.C.'s which can be resolved are those obtained from the previous set by making the transformation $x \to x^{-r}$.

7. Lie Theory

We now determine the general classes of D.E.'s which are invariant under specified groups of transformations and relationships among them. This leads to some rather simple functional equations.[7] If

$$y'' = F(x, y, p) \qquad (p = y') \tag{36}$$

is to be invariant under the one-parameter group of transformations

$$x = \lambda x_1, \qquad y = \lambda^{\alpha} y_1 \qquad (\alpha\text{-fixed}), \tag{37}$$

[7] This section is essentially not new and has been treated previously by Lie in his theory of one-parameter groups [13].

then F must satisfy the equation

$$F(\lambda x, \lambda^\alpha y, \lambda^{\alpha-1}p) \equiv \lambda^{\alpha-2}F(x, y, p)$$

for all λ, x, y, p. This is solved in the manner similar to establishing Euler's theorem on homogeneous functions assuming differentiability of F. Differentiating partially with respect to λ and setting $\lambda = 1$, we obtain the following necessary condition on F (a Lagrange linear P.D.E.):

$$x\frac{\partial F}{\partial x} + \alpha y\frac{\partial F}{\partial y} + (\alpha - 1)p\frac{\partial F}{\partial p} = (\alpha - 2)F. \tag{38}$$

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{\alpha y} = \frac{dp}{(\alpha - 1)p} = \frac{dF}{(\alpha - 2)F}.$$

Thus,

$$F = x^{\alpha-2}G\left(\frac{y}{x^\alpha}, \frac{y'}{x^{\alpha-1}}\right), \qquad (G\text{–arbitrary}) \tag{39}$$

is the general integral of (38) and provides all solutions which are not of the type called special [12].

For the class of n-th-order D.E.'s which are invariant under the transformation (37), we would find that similarly,

$$y^{(n)} = x^{\alpha-n}G\left(\frac{y}{x^\alpha}, \frac{y'}{x^{\alpha-1}}, \frac{y''}{x^{\alpha-2}}, \ldots, \frac{y^{(n-1)}}{x^{\alpha-n+1}}\right). \tag{40}$$

If (36) is to be invariant under the two-parameter group of transformations

$$x = \lambda x_1, \qquad y = \mu y_1, \tag{41}$$

we would then have to satisfy

$$F\left(\lambda x, \mu y, \frac{\mu p}{\lambda}\right) \equiv \mu\lambda^{-2}F(x, y, p).$$

This leads to the pair of P.D.E.'s

$$x\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial p} = -2F, \tag{42}$$

$$y\frac{\partial F}{\partial y} + p\frac{\partial F}{\partial p} = F. \tag{43}$$

Solving (42):

$$F = x^{-2}G(xp, y).$$

Substituting back into (43), we find that G must be homogeneous of first-order in xp, y. Thus,

$$y'' = \frac{y}{x^2} H\left(\frac{xy'}{y}\right), \qquad (H\text{-arbitrary}).$$

Proceeding in the same way, it follows that for a pair of simultaneous D.E.'s in y and z of order m, n, respectively, to be invariant under the two-parameter group

$$x = \lambda x, \qquad y = \mu y_1, \qquad z = \lambda^{\alpha}\mu^{\beta} z_1 \quad (\alpha, \beta\text{-fixed}), \tag{45}$$

the equations must have the form,

$$F_i\left(\frac{xy'}{y}, \frac{x^2y''}{y}, \ldots, \frac{x^m y^{(m)}}{y}, \frac{x^{-\alpha}z}{y^{\beta}}, \frac{x^{1-\alpha}z'}{y^{\beta}}, \ldots, \frac{x^{n-\alpha}z^{(n)}}{y^{\beta}}\right) = 0, \tag{46}$$

where F_1, F_2 are arbitrary functions. If the pair of simultaneous equations are to be invariant under the three-parameter group

$$x = \lambda x, \qquad y = \mu y_1, \qquad z = \eta z_1, \tag{47}$$

the equations must have the form

$$F_i\left(\frac{xy'}{y}, \frac{x^2y''}{y}, \ldots, \frac{x^m y^{(m)}}{y}, \frac{xz'}{z}, \frac{x^2z''}{z}, \ldots, \frac{x^n z^{(n)}}{z}\right) = 0.$$

Similar results can be obtained for systems of simultaneous equations which are invariant under a p-parameter group of linear homogeneous transformations.

We now consider nonlinear one-parameter groups of transformations. If the D.E.

$$y^{(n)} = F(y, y', \ldots, y^{(n-1)}, x) \tag{48}$$

is to be invariant under the transformation,

$$x = x_1\varphi(\lambda), \qquad y = y_1 + \lambda \qquad (\varphi(0) = 1), \tag{49}$$

then we must have

$$F\left(y + \lambda, \frac{y'}{\varphi(\lambda)}, \frac{y''}{\varphi(\lambda)^2}, \ldots, \frac{y^{(n-1)}}{\varphi(\lambda)^{n-1}}, x\varphi(\lambda)\right) \equiv \phi(\lambda)^{-n} F(y, y', \ldots, y^{(n-1)}, x).$$

As before, the D.E. must then be necessarily of the form.

$$y^{(n)} = x^{-n} G(xy', x^2y'', \ldots, x^{n-1}y^{(n-1)}, xe^{-y\varphi'(0)}). \tag{50}$$

Previously, the necessary condition was also sufficient. Here it will be also only if

$$\varphi(\lambda)\, e^{-\lambda\varphi'(0)} \equiv 1$$

or else the term $xe^{-y\varphi'(0)}$ does not appear in the equation. In the former case, $\varphi(\lambda) = e^{a\lambda}$ and we get the spiral group of transformations considered by Na [5, p. 87] in a more general context. In the latter case, the D.E. is also the general one invariant under the two-parameter group

$$x = x_1\mu, \qquad y = y_1 + \lambda.$$

If instead of (49), we had

$$x = x_1 + \lambda, \qquad y = y_1\varphi(\lambda) \qquad (\varphi(0) = 1), \tag{51}$$

then the D.E. will have to be of the form

$$y^{(n)} = yG\left(\frac{y'}{y}, \frac{y''}{y}, \dots, \frac{y^{(n-1)}}{y}, ye^{-x\varphi'(0)}\right). \tag{52}$$

Then as before either $\varphi(\lambda) = e^{a\lambda}$ or else the term $ye^{-x\varphi'(0)}$ does not appear in the equation.

If we had the two-parameter translation group

$$x = x_1 + \mu, \qquad y = y_1 + \lambda, \tag{53}$$

the corresponding D.E. is then

$$y^{(n)} = F(y', y'', \dots, y^{(n-1)}), \tag{54}$$

whereas if we had the one-parameter translation group

$$x = x_1 + \lambda, \qquad y = y_1 + a\lambda \quad (a\text{-fixed}), \tag{55}$$

the corresponding D.E. is

$$y^{(n)} = F(y', y'', \dots, y^{(n-1)}, y - ax). \tag{56}$$

Again, similar results can be obtained for systems of D.E.'s.

We now show that some of the previous types of D.E.'s and their associated group of transformations are equivalent in that they can be transformed into one another by a transformation of either the dependent or independent variable or both.

If in (50) which is invariant under (49) for $\varphi(\lambda) = e^{a\lambda}$, we let $y = \log z$, we get that

$$D^n \log z = x^{-n} G\left(xD \log z, \dots, x^{n-1}D^{n-1} \log z, \frac{z}{x^{\alpha}}\right) \tag{56}$$

is invariant under

$$x = x_1\mu, \qquad z = z_1\mu^{\alpha}.$$

(Here, we have let $\mu = e^{a\lambda}$ and $\alpha = 1/a$). That (56) is directly equivalent to (40) follows by some elementary operations (of course the two G's are different). For example, if $n = 3$, (56) becomes

$$x^3\left\{\frac{z'''}{z} - \frac{3z''z'}{z^2} - \frac{2z'^3}{z^3}\right\} = G\left(\frac{xz'}{z}, x^2\left(\frac{z''}{z} - \frac{z'^2}{z^2}\right), \frac{z}{x^{\alpha}}\right),$$

$$x^3\left\{\frac{z'''}{z} - \frac{3z''z'}{z^2} - \frac{2z'^3}{z^3}\right\} = G_1\left(\frac{xz'}{z}, \frac{x^2z''}{z}, \frac{z}{x^{\alpha}}\right),$$

$$\frac{x^3z'''}{z} = G_2\left(\frac{xz'}{z}, \frac{x^2z''}{z}, \frac{z}{x^{\alpha}}\right), \qquad \frac{x^3z'''}{z} = \frac{x^{\alpha}}{z}G_3\left(\frac{z}{x^{\alpha}}, \frac{z'}{x^{\alpha-1}}, \frac{z''}{x^{\alpha-2}}\right).$$

Similarly, some of the other equivalences are

(50) → (56) by letting $x = e^w$;

(52) → (41) by letting $x = \log w$;

(41) → (56) by letting $y = e^z, \quad x = e^w$.

It is known by the Lie theory of one-parameter groups that if a D.E. is invariant under some known group of one-parameter transformations, then one can depress the order of the equation [13]. For example, in Eq. (56), by letting $y = ax + z$, the equation is reduced to one only containing the dependent variable z and consequently can be reduced in order by one. Since Eqs. (41), (50), and (52) can be transformed into (56) as mentioned above, they are also reducible. Since Eq. (54) is obviously reducible in order by two, so is Eq. (44). Unfortunately, these possible reductions of order do not seem to be helpful here from a numerical standpoint.

Appendix

In a recent paper, Taylor [14] considering the equilibrium equation for two neighboring drops at different potentials, derived the necessary B.V.P.

$$\begin{gathered} y'' + x^{-1}y' = \alpha + \beta y^{-2}, \\ y'(0) = 0, \qquad y(1) = 1. \end{gathered} \tag{57}$$

Although cases were treated where $\alpha\beta \neq 0$, we will restrict our attention to the case $\alpha = 0$. In this case, the D.E. is invariant under a one-parameter group of transformations.

Taylor obtains solutions numerically which are not complete. Subsequently Ackerberg [15] obtained further solutions, arising from the nonuniqueness for a certain range of β. The method employed was to use a transformation used by Jones[8] in reducing Emden's equation to a first order one, i.e., by letting

$$u = \frac{x}{y}\frac{dy}{dx}, \qquad v = \frac{\beta x}{2y^2}\left(\frac{dy}{dx}\right)^{-1},$$

he obtains

$$\frac{du}{dv} = \frac{u(2v - u)}{2v(1 - u - v)} \tag{58}$$

where the B.C. at $x = 0$ is now $u = 0$, $v = 1$ and the B.C. at $x = 1$ requires that $uv = \beta/2$ (for a given β, the integral curve $u(v)$ must terminate on a given hyperbola). It follows that the point $u = \frac{2}{3}$, $v = \frac{1}{3}$ is a singular spiral point and is the terminus of the integral curve $u(v)$. The nonuniqueness for (57) follows from the existence of this spiral point. The numerical integration of (57) was carried out by first transforming (58) by means of polar coordinates centered at the spiral point and then using a fourth-order Runge-Kutta method. The following graph, relating the possible $y(0)$ values as a function of β, was obtained [15, p. 134] (Taylor's solutions correspond to the cases $y(0) > 0.5556$).

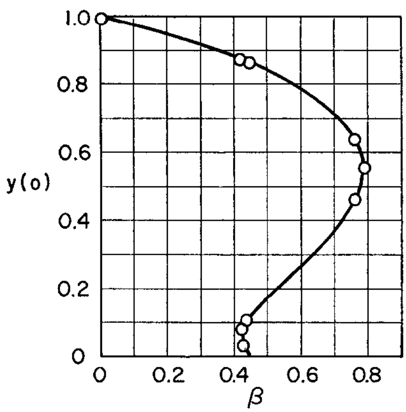

FIGURE 1

(The encircled points are the ones obtained here.)

[8] The equation can also be reduced in order by the methods just discussed.

The relation of $y(\text{o})$ vs β can be determined more simply using the one-parameter group associated with (57). This by no means is to deprecate the work of Ackerberg. By his reduction of (57) to (58) which he showed had a spiral singular point, he was immediately able to infer the non-uniqueness of the B.V.P. This problem illustrates a possible pitfall in solving similar B.V.P.'s by guessing values of $y(\text{o})$ and the correcting by some interpolation scheme. Unless the range of values guessed for $y(\text{o})$ is taken sufficiently large, one may not find all the possible solutions.

Before preceding with our conversion to an I.V.P., it is numerically very worthwhile to remove β from (57) by means of a scale transformation.[9] Letting, $y = z\beta^{1/3}$, (57) becomes

$$\begin{aligned} z'' + x^{-1}z' &= z^{-2}, \\ z'(\text{o}) = 0, \qquad z(1) &= \beta^{-1/3}. \end{aligned} \tag{59}$$

If $F(x)$ also satisfies the D.E., then z can be written in the form,

$$z = \lambda^{-2/3}F(\lambda x),$$

where λ is an arbitrary parameter. Then $F'(\text{o}) = 0$ and letting $F(\text{o}) = 1$, we have

$$z(\text{o}) = \lambda^{-2/3}$$

where

$$z(1) = \beta^{-1/3} = \lambda^{-2/3}F(\lambda).$$

To determine $z(x)$ [and also $y(\text{o})$], we solve the I.V.P. for $F(x)$ and check where $F(x) = x^{2/3}\beta^{-1/3}$ for any number of values of β. It is to be noted that only one I.V.P. need be solved and no guess work is needed. The I.V.P. was solved very simply here using a modified Euler method with $F' = P$, $\Delta x = 0.01$ on a time sharing console using "basic." The recurrence equations are

$$P_{n+1} = P_n\left(1 - \frac{1}{n}\right) + 0.01\,F_n^{-2},$$

$$F_{n+1} = F_n + 0.005(P_n + P_{n+1}),$$

where

$$P_2 = 0.01, \qquad F_2 = 1.0001.$$

Since (59) is singular at the origin, the starting pair (P_2, F_2) was obtained from the Taylor expansion of the solution

$$F = 1 + \frac{x^2}{4} - \frac{x^4}{32} - \frac{x^6}{324} + \cdots$$

[9] This was also done by Taylor and also for the example on p. 320.

(only the first two terms were needed). The reason for the modified Euler method rather than the unmodified version was to get the first few iterations to agree with the Taylor expansion. The machine was programmed to print out

$$n, F_n, \beta_i^{-1/3}(0.01n)^{2/3} \qquad (i = 1, 2, \dots, r)$$

whenever a pair of consecutive iterates produced a change of sign in any one of the expressions

$$\{F_n - \beta_i^{-1/3}(0.01n)^{2/3}\}, \qquad i = 1, 2, \dots, r.$$

At this stage, we were only interested in determining $y(0)$ which are given in the following table:

β	$y(0)$	$y(0)$ (Ackerberg)
0.77295	0.626	0.62733
	0.483	0.47935
0.425	0.869	0.86938
	0.089	0.0890
	0.032	0.03187
0.444	0.862	
	0.116	

Although, there should have been at least one more value of $y(0)$ for $\beta = 0.444$, the cutoff instruction to the machine was reached before it was obtained.

Considerably more accuracy could be obtained if a fourth-order Runge-Kutta method was used instead of the simpler modified Euler method. This would be necessary if one wanted to obtain the multiple solutions in the neighborhood of the singular point $\beta = \frac{4}{9}$.

The singular value $\beta = \frac{4}{9}$ appears in the particular solution of the D.E., i.e., $(\frac{4}{9})^{-1/3} x^{2/3}$ and it is likely that asumptotically

$$F(x) \approx (\tfrac{4}{9})^{-1/3} x^{2/3} + G(x)$$

where $G(x) = o(x^{2/3})$. In this case, since the number of solutions of

$$F(x) = \beta^{-1/3}x^{2/3}$$

increases without bound as $\beta \to \frac{4}{9}$, $G(x)$ will have to asymptotically oscillate about the value zero.

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